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Probabilistic logic metric and its application to approximate reasoning

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Abstract. The concepts of Λ-probabilistic truth degree and Λ-uncertainty degree are introduced in the paper. The conclusion that Λ-probabilistic truth degree satisfies Kolmogorov axioms is reached by discussing some of their properties. The paper proves that the Λ-uncertainty degree of conclusion is less than or equal to the sum of the product of Λ-uncertainty degree of each premise and its essentialness degree in a formal inference. The Λ-similarity degree and probabilistic logic pseudo-metric between formulas are introduced by using the Λ-uncertainty degree of formulas, and it indicates that there are not isolated points in the probabilistic logic pseudo-metric space under some conditions. As an application, proposals of two different approximate reasoning models in the probabilistic logic pseudo-metric space are raised as well as some examples to illustrate the practical application of these approximate reasoning models.

Key words. Probabilistic valuation, Λ-probabilistic degree, Λ-uncertainty, probabilistic logic pseudo-metric, approximate reasoning.

1. Introduction

In dealing with combination of logic deduction with numerical computing, probabilistic methods of logic deduction is extensively used. Because it realized to be the exact means featuring human thinking of which logic deduction and numerical estimation are naturally mixed [1–5]. Guojun Wang devoted to the introduction of grades within the framework of propositional logic systems and then establishing a kind of quantitative logic semantically. Truth degree plays a key role in studying quantitative logic. Many scholars focus on studying this concept and have proposed a variety of different truth degrees of propositions [6, 7]. Reference [8] introduces the concept of D-random truth degree, but the concept of D-random truth degree also has not reflected fundamentally the idea that the atomic formula takes ran-

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domly value in its value domain. In approximate reasoning, we should consider fully the randomness of every atomic formula taking value in value domain and apply random mathematical method to study quantitative logic, and then establish the random quantitative logic.

The rest of this paper is organized as follows. Section 2 introduces the concept of probabilistic valuation of proposition logic. Section 3 introduces the probabilistic truth degree based on the partial probabilistic valuation set and discusses its properties. Section 4 discusses the probabilistic truth degree based on finite probability space. Section 5 introduces the probabilistic truth degree based on the partial probabilistic valuation set and establishes probabilistic logic pseudo-metric between formulas. Section 6 presents two diverse approximate reasoning models and gives some reasoning instance. Section 7 concludes the study.

2. Probability semantic of propositional logic

Let $S = \{q_1, q_2, \cdots\}$, $\neg$ and $\to$ be two logic connectives and $F(S)$ be the free algebra of type $(\neg, \to)$ generated by S. The element in S is called atomic formula and the element in $F(S)$ is called formula. Suppose that Ω is a non-empty set and A is a σ-algebra on Ω. For any $\alpha,\ \beta \in A$, if we denote $\neg\alpha = -\alpha$, $\alpha \to \beta = (\Omega - \alpha) \cup \beta$, then $(\Omega,\ A)$ is also an algebra of type $(\neg, \to)$. Further, we assume that P is a probability measure on A, i.e., $(\Omega,\ A,\ P)$ is a probability space, in this case, a subset of Ω is also called an event.

Definition 2.1

(1) Let Ω be a non-empty set, (Ω, A) be a σ-algebra and P is a probability measure on A. The mapping v: $F(S) \to A$ is called an event valuation of $F(S)$ if v is a $(\neg, \to)$-type algebra homeomorphism, i.e., $\forall A,\ B \in F(S)$, $v\neg A) = \neg v(A)$ and $v(A \to B) = v(A) \to v(B)$. The probability $P(v(A))$, which can be viewed as a probabilistic truth value of A, under the event valuation v satisfies the relations

$$P(v(\neg A)) = 1 - P(v(A))$$

and

$$P(v(A \to B)) = P(v(A) \to v(B)) = P((\Omega - v(A)) \cup v(B) =$$
$$= P((\Omega - v(A)) \cup (v(A) \cap v(B)) = 1 - P(v(A)) + P(v(A) \cap v(B))\,.$$

The event valuation with probability truth value of formulas is called probabilistic valuation of formulas and the set of all probability valuations of $F(S)$ is denoted by Σ_{P}.

(2) Let $A \in F(S)$. If $\forall v \in \Sigma_{\mathrm{P}}\ P(v(A)) = 1$, then A is called the probabilistic tautology. If $\forall v \in \Sigma_{\mathrm{P}}$, $P(v(A)) = 0$, then A is called the probabilistic contradiction.

Obviously, for any probabilistic valuation v, we have

$$P(v(A \vee B)) = P(v(A) \cup v(B))$$

and

$$P(v(A \wedge B)) = P(v(A) \cap v(B))\,,$$

where v is determined uniquely by its restriction $v|_S$ to S because $F(S)$ is the free algebra generated by S.

Definition 2.2

Let A, $B \in F(S)$. If $A \to B$ is a probabilistic tautology, then we call A probability logic implies B (for short, A implies B), denoted it by $A \Rightarrow B$. If $A \Rightarrow B$ and $B \Rightarrow A$, then we call A and B probability logic equivalence (for short, A and B equivalence), denoting it as $A \Leftrightarrow B$.

Proposition 2.3

(1) $\forall v \in \Sigma_{\mathrm{P}},\ 0 \leq P(v(A)) \leq 1\,.$

(2) If $A \Rightarrow B$, then

$$\forall v \in \Sigma_{\mathrm{P}} \quad P(v(A \to B)) = 1 - P(v(A)) + P(v(A) \cap v(B)) = 1\,,$$

i.e.

$$P(v(A)) = P(v(A) \cap v(B))\,.$$

Hence,

$$P(v(A)) = P(v(A) \cap v(B)) \leq P(v(B))\,,$$

$$P(v(A) \cap v(B)) = P(v(A)) = \min\{P(v(A)),\, P(v(B))\}\,,$$

$$P(v(A) \cup v(B)) = P(v(B)) = \max\{P(v(A)),\, P(v(B))\}\,.$$

(3) If $A \Leftrightarrow B$, then $\forall v \in \Sigma_{\mathrm{P}}\ P(v(A)) = P(v(B))\,.$

(4) If A and B are logically incompatible, i.e., $A \wedge B$ is a contradiction, then

$$P(v(A \vee B)) = P(v(A) \cup P(v(B)) = P(v(A)) + P(v(B)) - P(v(A)) \cap P(v(B)) =$$

$$= P(v(A)) + P(v(B)) - P(v(A)) \wedge P(v(B)) = P(v(A)) + P(v(B))\,.$$

(5) If $\forall v \in \Sigma_{\mathrm{P}}$, then $v(A)$ and $v(B)$ are independent and we call them independent. When $v(A)$ and $v(B)$ are independent, we have

$$P(v(A \wedge B)) = P(v(A) \cap v(B)) = P(v(A)) \times P(v(B))$$

and

$$P(v(A \vee B)) = P(v(A) \cup P(v(B)) = P(v(A)) + P(v(B)) - P(v(A)) \times P(v(B)).$$

Proposition 2.3 shows that $P(v(\cdot))$ satisfies Kolmogorov axioms [9], hence $P(v(\cdot))$ can be viewed as a probability measure on $F(S)$.

3. Probabilistic truth degree of logic formulas

We have already known every probabilistic valuation v: $F(S) \to \Gamma$ is uniquely determined by its restriction $v|_S$ to S, in other words, every mapping v: $S \to \Gamma$ can uniquely be extended to a probabilistic valuation. If $v(q_k) = v_k$, $k = 1, 2, \cdots$, then $T(v) = (v_1, v_2, \cdots) \in \prod_{k=1}^{\infty} \Gamma_k$ is called a state-description of S, where $\Gamma_k = \Gamma$ and $\prod_{k=1}^{\infty} \Gamma_k$ is not viewed as the usual infinite product of σ-algebra but is viewed as a direct product of sets Γ_k, $k = 1, 2, \cdots$, and it is also denoted by Γ^{∞}. Conversely, if $(v_1, v_2, \cdots \in \prod_{k=1}^{\infty} \Gamma_k$, then there exists a unique probabilistic valuation $v \in \Sigma_{\mathrm{P}}$ such that $v(q_k) = v_k$, $k = 1, 2, \cdots$. Hence $\phi: \ \Sigma_{\mathrm{P}} \to \prod_{k=1}^{\infty} \Gamma_k$ and $\phi(v) = T(v)$ is a bijection.

Suppose that Γ^* is a σ-algebra on Γ^{∞} and μ^* is a probability measure on Γ^*. The measure μ^* on Γ^* can be transferred into μ on Σ_{P} by means of φ, i.e., for any $\Sigma \subseteq \Sigma_{\mathrm{P}}$, if $\varphi(\Sigma) \in \Gamma^*$, then $\mu(\Sigma = \mu^*(\varphi(\Sigma))$, and μ is called the induced probability measure by μ^*. If we denote $\Gamma = \{\Sigma | \Sigma \subseteq \Sigma_{\mathrm{P}},\ \varphi\Sigma \in \Gamma^*\}$, then $(\Sigma_{\mathrm{P}}, \Gamma, \mu)$ is a probability measure, which it is also called the induced probability measure space by Γ^{∞}, (Γ^*, μ^*).

According to the view in [5, 6], a formula A determines uniquely a function on probabilistic valuation set Σ_{P}:

$$A : \Sigma_{\mathrm{P}} \to [0,1], \quad A(v) = \mu(v(A))\,.$$

Definition 3.1

Suppose that A^* is a $\sigma-$ algebra on Γ^{∞}, μ^* is a probability measure on Γ^* and $(\Sigma_{\mathrm{P}}, \Gamma, \mu)$ is the induced probability measure space by $(\Gamma^{\infty}, \Gamma^*, \mu^*)$. Let $A \in F(S)$, $\Lambda \subseteq \Sigma_{\mathrm{P}}$ be a $\mu-$measurable set, and define

$$\tau_{\Lambda}(A) = \int_{\Lambda} A(v)\,\mathrm{d}\mu = \int_{\Sigma_{\mathrm{P}}} A(v)\chi_{\Lambda}(v)\,\mathrm{d}\mu\,.$$

Then $\tau_{\Lambda}(A)$ is called the $\Lambda-$ probabilistic truth degree of A. If $\Lambda = \Sigma_{\mathrm{P}}$, then $\tau_{\Sigma_P}(A)$, briefly $\tau(A)$, is called a probabilistic truth degree of A.

Remark 3.2

Let $A = A(q_1, q_2, \ldots, q_t)$ be a formula built up from atomic formulas with $q_1, q_2, \ldots, q_t$. Obviously, formula A determines a function with t variables

$$\bar{A}^{(t)} : \Gamma^t \to [0,1]$$

and

$$\bar{A}(v_1, v_2, \ldots, v_t) = \mu(\varphi^{-1}((v_1, v_2, \ldots, v_t) \times \prod_{k=t+1}^{\infty} \Gamma_k)(A)) = \mu(v(A))\,,$$

where $\Gamma^t = \prod_{k=1}^{t} \Gamma_k$. Let

$$C^t = \{E | E \subseteq \Gamma^t,\ E \times \prod_{k=t+1}^{\infty} \Gamma_k \in \Gamma^*\}\,.$$

Then C^t is a $\sigma-$algebra on Γ^t.

Hence if we define $\mu^*(t) : \Gamma^t \to [0, 1]$

$$\mu^*(t)(E) = \mu^*(E \times \prod_{k=t+1}^{\infty} \Gamma_k),\ E \in C^{(t)}\,,$$

then $\mu^*(t)$ is a probability measure on Γ^t, called the restriction of μ^* on Γ^t. Then we can obtain the following computing equation of probabilistic truth degree

$$\tau_\Lambda(A) = \int_{\varphi(\Lambda)} \overline{A}^{(t)}(\cdot)\,\mathrm{d}\mu^*(t).$$

Conversely, let P be a probability measure on Γ^t and μ_m the probability measure on $\Gamma_m = \Gamma(m = t+1, t+2, \cdots)$. Because Γ^∞ can also be viewed as the infinite product space of $\Gamma^t, \Gamma_{t+1}, \Gamma_{t+2}, \cdots$, then $P, \mu_{t+1}, \mu_{t+2}, \cdots$ determine a unique product probability measure μon Γ^∞. It is easy to check that $\mu^*(t) = P$.

For a formula $A = A(q_1, q_2, \cdots q_t)$ with t atomic formulas, t variables function $\overline{A}(v_1, v_2, \cdots, v_t)$ can also be viewed as $t+i$ variables function: $\overline{A}^{(t+i)}(v_1, \cdots, v_t, v_{t+1}, \cdots v_{t+i}) = \overline{A}(v_1, v_2, \cdots v_t)$.. Hence probabilistic truth degree have the following integral form invariant property.

Proposition 3.3

Let $A = A(q_1, q_2, \cdots q_t)$ be a formula with t atomic formulas. Then

$$\tau_\Lambda(A) = \int_{\varphi(\Lambda)} \overline{A}^{(t)}(\cdot)\,\mathrm{d}\mu^*(t) = \int_{\varphi(\Lambda)\times\Gamma^i} \overline{A}^{(t+i)}(\cdot)\,\mathrm{d}\mu^*(t+i)\,.$$

Proposition 3.4

Probabilistic truth degree have the following properties:

1. $0 \le \tau_\Lambda(A) \le 1$.

2. If $A \Leftrightarrow B$, then $\tau_\Lambda(A) = \tau_\Lambda(B)$.

3. If A is a probabilistic tautology (contradiction), then $\tau_\Lambda(A) = 1(\tau_\Lambda(A) = 0)$.

4. $\tau_\Lambda(\neg A) = 1 - \tau_\Lambda(A)$.

5. $\tau_\Lambda(\neg A) = 1 - \tau_\Lambda(A)$.

6. If $A \to B$ is a probabilistic tautology, then $\tau_\Lambda(A) \le \tau_\Lambda(B)$.

Definition 3.5

We call $U_\Lambda(A) = 1 - \tau_\Lambda(A) = \int_\Lambda (1 - A(v))\,\mathrm{d}\mu$ the Λ-uncertainty degree of formula A. If $\Lambda = \Sigma_P$, then $U_{\Sigma_P}(A)$, briefly U(A), is called the uncertainty degree of A.

Definition 3.6

Let $\{A_1, A_2, \cdots A_n\} \subseteq F(S), A^* \in F(S)$. If $\{A_1,\ A_2, \cdots A_n\} \mapsto A^*$, then "$\{A_1, A_2, \cdots A_n\} \Rightarrow A^*$" is called an effective reasoning, $A_1, A_2, \cdots A_n$are called premises of reasoning and A^* is called a conclusion of reasoning. For an effective reasoning "$\{A_1, A_2, \cdots A_n\} \Rightarrow A^*$", some premises in it may be necessary, i.e., conclusion A^* may be the conclusion of a proper subset $\{A_{i1}, \cdots A_{ik}\}(k < n)$ of $\{A_1, A_2, \cdots A_n\}$. In order to distinguish the necessity of a premise in an effective reasoning, Nillson introduced a concept of essentialness degree of premise [3].

Definition 3.7

Let $\{A_1, A_2, \cdots A_n\} \Rightarrow A^*$ be an effective reasoning and $F = \{A_1, A_2, \cdots A_n\}$. For a subset E of F, if conclusion A^* is not induced from premises $F - E$ then E is called an essential premise. The number of minimum premises set, which the number of premise is minimal in all essential premises set containing premise A_i, is denoted by $\delta(A_i)$. We call $e(A_i) = 1/\delta(A_i)$ an essentialness degree of A_i. If there is not a minimum premises set containing premise A_i, then $e(A_i) = 0$.

Theorem 3.8

Let $\{A_1, A_2, \cdots A_n\} \Rightarrow A^*$ be an effective reasoning. Then the uncertainty degree of conclusion is less than or equal to the sum of the product of uncertainty degree of every premise and its essentialness degree.

Proof

Denote $\Psi = \{A_1, A_2, \cdots A_n\}$. By the integral form invariant properties of probabilistic truth degree, we may assume that the all formulas in Ψ have t atomic formulas. Let t and the value state be $T(v)$. Note that for every probabilistic valuation v, $\mu(v(\cdot))$ can be viewed as a probability on Ψ, the uncertainty degree of formulas $A_1, A_2, \cdots A_n, A^*$ are $1 - \mu(v(A_1)), 1 - \mu(v(A_2)), \cdots,\ 1 - \mu(v(A_n)),\ 1 - \mu(v(A^*))$, respectively. By probability logic fundamental theorem [3] we have

$$1 - \mu(v(A^*)) \le e(A_1)(1 - \mu(v(A_1))) + e(A_2)(1 - \mu(v(A_2))) \cdots e(A_n)(1 - \mu(v(A_n)))\,.$$

Then

$$\int_\Lambda (1 - A^*(v))\,\mathrm{d}\mu \le \int_\Lambda e(A_1)(1 - A_1(v)n)\,\mathrm{d}\mu + \int_\Lambda e(A_2)(1 - A_2(v))\,\mathrm{d}\mu +$$

$$+ \cdots + \int_\Lambda e(A_n)(1 - A_n(v))\,\mathrm{d}\mu = e(A_1)\int_\Lambda (1 - A_1(v)n)\,\mathrm{d}\mu +$$

$$+ e(A_2(v)\int_\Lambda (1 - A_2(v))\,\mathrm{d}\mu + \cdots + e(A_n)\int_\Lambda (1 - A_n(v))\,\mathrm{d}\mu\,.$$

Hence

$$U_\Lambda(A^*) \leq e(A_1)U_\Lambda(A_1) + e(A_2(v))U_\Lambda(A_2) + \cdots + e(A_n)U_\Lambda(A_n)\,.$$

4. Probabilistic truth degree based on finite probability space

Let $\Omega = \{\Omega_1, \Omega_2, \cdots, \Omega_n\}$ and Γ(σ-algebra) the set of all subsets of Ω. Suppose that $A = A(q_1, q_2, \cdots q_t)$ and $Auto(A) = \{q_1, q_2, \cdots q_t\}$ is the set of all atomic formulas in A. Then there are 2^{nm} different probabilistic valuation on A, i.e., $|\Sigma_P| = 2^{nm}$. For $v \in \Sigma_P$ denote $T_A(v) = (v(q_1), \cdots v(q_n)) \in \Gamma^n$, the set of all value states is denoted by $T_A = \{T_A(v_1), T_A(v_2), \cdots, T_A(v_l)\}(l = 2^{nm})$. Suppose that P_n is a normal probability distribution on T_A, i.e., $0 < P_n(T_A(v_i)) < 1(i = 1, 2, \cdots l), \sum_{i=1}^{l} P_n(T_A(v_i)) = 1$. Hence we obtain

$$\tau_\Lambda(A) = \int_{\varphi(\Lambda)} \overline{A}^{(t)}(\cdot)\,\mathrm{d}\mu^*(t) = \sum_{i=1}^{l} P(v_i(A))P_n(T_A(v_i))\,.$$

In particular, if P_n is an uniform probability distribution on T_A, i.e., $P_n(T(v_i)) = 1/2^{nm}$ we also obtain

$$\tau_u = \frac{1}{2^{nm}} \sum_{i=1}^{2^{nm}} P(v_i A))\,.$$

In the following we discuss in detail the properties of $\tau_u(A)$. Obviously, we have $0 \leq \tau_u(A) \leq 1$ for any $A \in F(S)$.

Proposition 4.2

Suppose that there is not common atomic formulas in A and B. Then $\tau_u(A \wedge B) = \tau_u(A) \times \tau_u(B)$.

Proof

It is no hurt to assume that $A = A(q_1, q_2, \cdots q_{n_1})$,$B = B(q_{n_1+1}, \cdots, q_{n_1+n_2})$. Let $S_A = \{q_1, \cdots, q_{n_1}\}$, $S_B = \{q_{n_1+1}, \cdots q_{n_1+n_2}\}$. Then each probabilistic valuation $v$of $A \wedge B$ determines corresponding a probabilistic valuation $v|_{S_A}$ of A and $v|_{S_B}$ of B. Hence

$$\tau_u(A \wedge B) = \frac{1}{2^{m(n_1+n_2)}} \sum_v [P(v(A) \cap v(B))] =$$

$$= \frac{1}{2^{m(n_1+n_2)}} \sum_{v|_{S_A}} [P(v\,|_{S_A}\,(A)) \cdot \sum_{v|_{S_B}} P(v\,|_{S_B}\,(B))] = \tau_u(A) \cdot \tau_u(B)\,.$$

Corollary 4.3

$\tau_u(q_1 \wedge q_2 \wedge \cdots \wedge q_n) = \frac{1}{2^n}$.

Proof

Let $\Omega = \{\Omega_1, \Omega_2, \cdots \Omega_m\}$. Then $\tau_u(q_1) = \frac{1}{2^m} \sum_v P(v(q_1))$. By computing we have $\sum_v P[v(q_1)] = 2^{m-1}$. Hence

$$\tau_u(q_1) = \frac{1}{2^m} \times 2^{m-1} = \frac{1}{2}\,.$$

Analogously,

$$\tau_u(q_2) = \cdots = \tau_u(q_n) = \frac{1}{2^m} \times 2^{m-1} = \frac{1}{2}\,.$$

It follows from Proposition 4.2 that $\tau_u(q_1 \wedge q_2 \wedge \cdots \wedge q_n) = \frac{1}{2^n}$.

Theorem 4.4

The set $\{\tau_u(A)\,|A \in F(S)\}$has no isolated points in [0,1].

Proof

Let $A = A(q_1, q_2, \cdots q_n) \in F(S), \varepsilon > 0$. In the following we prove that there exists a formula $B \in F(S)$ such that $|\tau_u(A) - \tau_u(B)| < \varepsilon$ and $\tau_u(A) \neq \tau_u(B)$.

1. $\tau_u(A) = 0$. If we take k such that $1/2^k < \varepsilon$, then by Proposition 4.3 there is $B(k) = q_{n+1} \wedge \cdots \wedge q_{n+k}$, such that $\tau_u(B(k)) = 1/2^k$. In this case, if we take $B = B(k)$, then $|\tau_u(A) - \tau_u(B)| = \tau_u(B(k)) = 1/2^k < \varepsilon$.
2. $\tau_u(A) = 1$. Let $B = \neg B(k)$, then $\tau_u(B) = 1 - \tau_u(B(k)) \neq \tau_u(A)$ and $|\tau_u(A) - \tau_u(B)| = 1/2^k < \varepsilon$.
3. $0 < \tau_u(A) < 1$. Let $B = A \vee B(k)$, then it follows from Propositions 3.4 and 4.2 that $\tau_u(B) = \tau_u(A) + \tau_u(B(k)) - \tau_u(A \wedge B(k)) = \tau_u(A) + \tau_u(B(k)) - \tau_u(A) \cdot \tau_u(B(k))$. Hence $\tau_u(A) \neq \tau_u(B)$ and

$$|\tau_u(A) - \tau_u(B)| = \tau_u(B(k))(1 - \tau_u(A)) < \tau_u(B(k)) = \frac{1}{2^k} < \varepsilon\,.$$

5. Probabilistic logic pseudo-metric between formulas

In order to meet the demands of approximate reasoning in practical application, as well as to the integrity of the theory, in the next section we introduce Λ-probabilistic truth degree based on the partial probabilistic valuation set Λ, and establish the Λ-pseudometric on the formulas set $F(S)$.

Definition 5.1.

Let $A, B \in F(S)$ and denote

$$\xi_\Lambda(A, B) = \tau_\Lambda((A \to B) \wedge (B \to A))\,.$$

Then $\xi_\Lambda(A, B)$ is called the Λ-resemblance degree between A and B. If $\xi_\Lambda(A, B) = 1$, then A and B are called Λ-resemblance. In particular, when $\Lambda = \Sigma_P$, the $\xi_{\Sigma_P}(A, B)$ is called the P-resemblance degree between A and B.

Theorem 5.2.

Let $A, B \in F(S)$. Then

1. $\xi_\Lambda(A,B) = \xi_\Lambda(B,A)$.

2. If A and B are logically equivalence, then A and B are Λ-resemblance.

Theorem 5.3.

Let $A, B, C \in F(S)$. Denote $\rho_\Lambda(A,B) = 1-\xi_\Lambda(A,B) = U_\Lambda((A \to B)\wedge(B \to A))$. Then

(1). $\rho_\Lambda(A,B) = 0$.

(2). $\rho_\Lambda(A,C) \le \rho_\Lambda(A,B) + \rho_\Lambda(B,C)$.

Proof

The proof of (1) is easy, in the following we prove (2). It is easy to check that

$$\{(A \to B) \wedge (B \to A), (B \to C) \wedge (C \to B)\} \Rightarrow (A \to C) \wedge (C \to A)$$

is an effective reasoning, and $e((A \to B)\wedge(B \to A)) = 1, e((B \to C)\wedge(C \to B)) = 1$. Hence by Theorem 3.8 we know that $U_\Lambda((A \to C)\wedge(C \to A)) \le U_\Lambda((A \to B)\wedge(B \to A)) + U_\Lambda((B \to C) \wedge (C \to B))$. Therefore, $\rho_\Lambda(A,C) \le \rho_\Lambda(A,B) + \rho_\Lambda(B,C)$.

Remark 5.4.

By Theorem 5.3, we know that $\rho_\Lambda(A,B)$ is a pseudo-metric on $F(S)$ and call $(F(S), \rho_\Lambda(A,B))$ a probabilistic logic pseudo-metric space. The probabilistic logic pseudo-metric with respective to probabilistic truth degree τ_0 and τ_u are denoted by ρ_u and ρ_0 respectively.

Theorem 5.5.

Suppose that τ_A satisfies the condition: $\tau_\Lambda(q_1 \wedge q_2 \wedge \cdots \wedge q_t) \to 0 (t \to \infty)$. Then there is no isolated points in probabilistic logic pseudo-metric space $(F(S), \rho_\Lambda)$.

Proof

Let $A = A(q_1, q_2, \cdots q_n)$ and $\varepsilon > 0$. In the following we prove that there exists $B \in F(S), B \ne A$ such that $\rho_\Lambda(A,B) < \varepsilon$. Since $\tau_\Lambda(q_1 \wedge q_2 \wedge \cdots \wedge q_t) \to 0(t \to \infty)$, we know that there exists a positive number k such that $\tau_\Lambda(q_{n+1} \wedge \cdots \wedge q_{n+k}) < \varepsilon$. Taking $C = \neg(q_{n+1} \wedge \cdots \wedge q_{n+k})$ and $B = A \wedge C$, then $(A \to B) \wedge (B \to A)$ and $A \to C$ are logically equivalence. Hence

$$\rho_\Lambda(A,B) = U_\Lambda((A \to B) \wedge (B \to A)) = U_\Lambda(A \to C) = 1 - \tau_\Lambda(A \to C)$$

and

$$\le 1 - \tau_\Lambda(C) = \tau_\Lambda(q_{n+1} \wedge \cdots \wedge q_{n+k}) < \varepsilon\,.$$

Obviously, $B \ne A$. It follows that B is the demanded formula. This prove that there are no isolated points in probabilistic logic pseudo-metric space $(F(S), \rho_\Lambda)$.

6. Some approximate reasoning models in $(F(S), \rho_\Lambda)$.

Now that the probabilistic logic pseudo-metric ρ_Λ has been introduced in $F(S)$, we are ready to provide two kind of approximate reasoning models in $F(S)$.

1. If $\inf\{\rho_\Lambda(A,B)\,|B \in D(\Psi)\} < \varepsilon$, then we say A is a I-type conclusion of Ψ with error less than ε with respect to Λ, denoted it by $A \in D^{(1)}_{\varepsilon,\Lambda}(\Psi)$. In particular, when $\Psi = \emptyset$ and $\Lambda = \Sigma_P$, we say that A is a I-type theorem with error less than ε, denoted it by $(1,\varepsilon) \mapsto_P A$.
2. If $\inf\{H(D(\Psi), D(\Xi))|\Xi\,|-A\} < \varepsilon$ where H is the Hausdorff distance on $2^{F(S)} - \{\emptyset\}$, then we say A is a II-type conclusion of Ψ with error less than ε with respect to Λ, denoted it by $A \in D^{(2)}_{\varepsilon,\Lambda}(\Psi)$. In particular, when $\Psi = \emptyset$ and $\Lambda = \Sigma_P$, we say that A is a II-type theorem with error less than ε, denoted it by $(2,\varepsilon) \mapsto_P A$.

Theorem 6.2.
Let $\Psi \subset F(S), A \in F(S), \varepsilon > 0$. Then $A \in D^{(1)}_{\varepsilon,\Lambda}(\Psi)$ if and only if

$$\sup\{\tau_\Lambda(B \to A)\,|B \in D(\Psi)\} > 1 - \varepsilon\,.$$

Proof
Suppose that $A \in D^{(1)}_{\varepsilon,\Lambda}(\Psi)$. Then there is $B_0 \in D(\Psi)$ such that $\rho_\Lambda(A, B_0) < \varepsilon$. Since

$$\rho_\Lambda(A,B_0) = 1 - \xi_\Lambda(A,B_0) = 1 - \tau_\Lambda((A \to B_0) \wedge (B_0 \to A)) \geq 1 - \tau_\Lambda(B_0 \to A)\,,$$

we have that

$$1 - \sup\{\tau_\Lambda(B \to A)\,|B \in D(\Psi)\} \leq 1 - \tau_\Lambda(B_0 \to A) < \varepsilon\,.$$

It follows that $\sup\{\tau_\Lambda(B \to A)\,|B \in D(\Psi)\} > 1 - \varepsilon$.

Conversely, let $\sup\{\tau_\Lambda(B \to A)\,|B \in D(\Psi)\} > 1 - \varepsilon$. Then $inf\{1 - \tau_\Lambda(B \to A)\,|B \in D(\Psi)\} < \varepsilon$. It is easy to check that $(A \to A \vee B) \wedge (A \vee B \to A) \sim B \to A$. Hence

$$\rho_\Lambda(A, A \vee B) = 1 - \xi_\Lambda(A, A \vee B) = 1 - \tau_\Lambda((A \to A \vee B) \wedge (A \vee B \to A)) =$$

$$= 1 - \tau_\Lambda(B \to A)\,.$$

This means that $\inf\{\rho_\Lambda(A, A \vee B)\,|B \in D(\Psi)\} < \varepsilon$. Thus there exists $B \in D(\Psi)$ such that $\rho_\Lambda(A, A \vee B) < \varepsilon$. Since $B \in D(\Psi)$, we have that $B \vee A \in D(\Psi)$. This shows that $\inf\{\rho_\Lambda(A,B)\,|B \in D(\Psi)\} \leq \rho_\Lambda(A, A \vee B) < \varepsilon$. Therefore $A \in D^{(1)}_{\varepsilon,\Lambda}(\Psi)$.

Theorem 6.3
Let $\Psi \subset F(S), A \in F(S), \varepsilon > 0$. If $A \in D^{(2)}_{\varepsilon,\Lambda}(\Psi)$. then $A \in D^{(??)}_{\varepsilon,\Lambda}(\Psi)$.
Proof

Let $A \in D^{(2)}_{\varepsilon,\Lambda}(\Psi)$. Then there exists $\Xi \subset F(S)$ such that $H(D(\Psi), D(\Xi)) < \varepsilon$ and $\Xi \mid\!- A$. It follows from $\Xi \mid\!- A$ that $B \in D(\Xi)$. Hence

$$\inf\{\rho_\Lambda(A, B) | B \in D(\Psi)\} = \rho_\Lambda(A, D(\Psi)) \leq H(D(\Psi), D(\Xi)) < \varepsilon\,.$$

Therefore $A \in D^{(1)}_{\varepsilon,\Lambda}(\Psi)$.

Example 6.4

Suppose that $\Omega = \{a, b, c\}, P(a) = \frac{1}{2}, P(b) = \frac{1}{4}, P(c) = \frac{1}{4}, \Gamma = 2^\Omega$ be the power set of Ω. Let $\Psi = \{q_1 \vee q_2, q_1 \vee q_3\}, A = q_1$. Then $\Psi \cup \{A\}$ has $2^9 = 512$ different value states $T_{\Psi\cup\{A\}} = \{T(v_1), T(v_2), \cdots T(v_{512})\}$, where the number of v_i is arranged as follows: we assign value to the q_1, q_2, q_3, according to firstly $\emptyset$, secondly $\{a\}, \{b\}, \{c\}$, again $\{a, b\}, \{a, c\}, \{b, c\}$, finally Ω.

For example, $v_1(q_1) = \emptyset, v_1(q_2) = \emptyset, v_1(q_3) = \emptyset$; $v_2(q_1) = \emptyset, v_2(q_2) = \emptyset, v_2(q_3) = \{a\}$; $v_3(q_1) = \emptyset, v_3(q_2) = \emptyset, v_3(q_3) = \{b\}; \cdots; v_{512}(q_1) = \Omega, v_{512}(q_2) = \Omega, v_{512}(q_3) = \Omega$. Suppose that there is a probability distribution $P_t$on $T_{\Psi\cup\{A\}}$:$P_t(T(v_i)) = \alpha(\frac{1}{2})^{i-1}$, where $\alpha = \frac{1}{2(1-(\frac{1}{2})^{512})} \approx \frac{1}{2}$.

Since $q_1 \vee q_2 \to (q_1 \vee q_3 \to q_1)$ is not a theorem, $q_1 \notin D(\Psi)$. Note that$(q_1 \vee q_2) \wedge (q_2 \vee q_3)$ and $q_1 \vee (q_2 \wedge q_3)$ are provable equivalence and $(q_1 \vee q_2) \wedge (q_2 \vee q_3) \in D(\Psi)$. Hence $q_1 \vee (q_2 \wedge q_3) \in D(\Psi)$, and

$$\tau(q_1 \vee (q_2 \wedge q_3) \to q_1) = \tau(q_2 \wedge q_3 \to q_1) =$$

$$= \tau(\neg q_2 \vee \neg q_3 \vee q_1) = 1 - \tau(\neg q_1 \wedge q_2 \wedge q_3)\,,$$

and

$$\tau(\neg q_1 \wedge q_2 \wedge q_3) = \int_{\sum p} (\neg q_1 \wedge q_2 \wedge q_3)(v)\,\mathrm{d}\mu =$$

$$= \sum_{i=1}^{512} P(v_i(\neg q_1 \wedge q_2 \wedge q_3) P_t(T(v_i)) = 0.0184\,.$$

Thus, for $\varepsilon = 0.02$, there exists a formula $q_1 \vee (q_2 \wedge q_3) \in D(\Psi)$ such that$\tau(q_1 \vee (q_2 \wedge q_3) \to q_1) > 1 - \varepsilon$. Therefore. $A = q_1$ is a I-type conclusion of Ψ with error less than 0.02.

Now we answer the approximate reasoning mentioned in Introduction.

At first, we symbolize the proposition in the reasoning.q_1: John will live in Warsaw on Dec. 21 next year, q_2: John will live in Athens on Dec. 21 next year, q_3: John will live in Vienna on Dec. 21 next year. Then premise set $\Psi = \{q_1 \vee q_2, q_1 \vee q_3\}$, conclusion $B = q_1$. We assume that the probability of events "John will live in Warsaw on Dec. 21 next year (denote it by a)", "John will live in Athens on Dec. 21 next year (denote it by b)", "John will live in Vienna on Dec. 21 next year (denote it by c)" are 0.5, 0.25, 0.25 respectively. By Example 6.4, A is a Γ conclusion in probability truth with error less than 0.02. Hence, based on the current facts we can approximatively reasoning that John will live in Warsaw on Dec. 21 next year, the reasoning error being less than 0.02.

7. Conclusion

This paper extend the classical two-valued proposition logic along the direction of the randomization. By extending the value domain $\{0, 1\}$ of propositional logic to a probability space we establish probabilistic semantics of propositional logic, the classical two-valued semantic of proposition logic is a special case of it. The concepts of Λ-probabilistic truth degree and Λ-uncertainty degree are introduced, and some properties of them are discussed. The conclusions show that Λ-uncertainty degree satisfies Kolmogorov axioms. It is proved that the set of probabilistic truth degree based on valuation set on independent events of all formulas has not isolated points in [0,1] and the Λ-uncertainty degree of conclusion is less than or equal to the sum of the product of Λ-uncertainty degree of every premise and its essentialness degree in a formal reasoning. The Λ-similarity degree and probabilistic logic pseudo-metric between formulas are introduced by using the Λ-uncertainty degree of formulas, and it is proved that there are not isolated points in the probabilistic logic pseudo-metric space. As an application, two different approximate reasoning models in the logic pseudo-metric space are proposed, and some examples to illustrate the practical application of these approximate reasoning models are given.

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